Physics 200A Fall 2013

I.) Basic Lagrangian Mechanics (Read L&L: Chapt 1, 2)

→ Principle of Least Action / Hamilton's Principle

If system of point particles (i.e. no internal degrees of freedom), such that

- parametrized by generalized coordinates $q_1, q_2 \ldots q_s$, and generalized velocities $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_s$ where $q_i = q_i(t)$

(n.b. time is parameter).

N.B. Generalized coordinates need not correspond to usual/familiar coordinate system.

- let system be described by a function $L = L(\underline{q}_i, \dot{\underline{q}}_i, t)$.

explicit time dependence possible

L is Lagrangian.

Then:

Trajectory $\underline{q}_1(t_1) \rightarrow \underline{q}_2(t_2)$ is one which minimizes action S

$$S = \int_{t_1}^{t_2} dt \; L(q_i, \dot{q}_i, t)$$

Action – functional of trajectory $q_i(t)$
– enables g.c.

i.e. trajectory selected by Principle of Least Action.

Observations:

- variational principle allows use of generalized coordinates

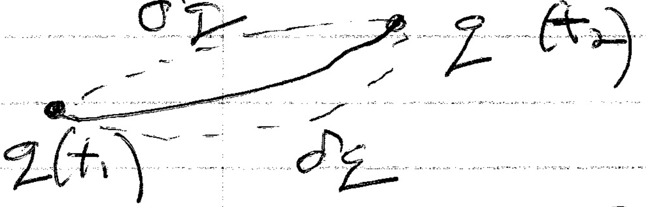
  i.e. S minimized, parametrization independent. S not calculated.

- S it-self not determined. EOM is determined.

- PLA is energy method, as $L = T - U$ (tbd)

Now, consider variation of (necessary, not sufficient, for minimization) S:

$$\delta S = [redacted] \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right)$$



exchange deriv.

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q) + \frac{\partial L}{\partial q} \delta q \right)$$

$$= \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\, \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)$$

but $\delta q(t_{1,2}) = 0$, i.e. on end points, so

$$\delta S = \int_{t_1}^{t_2} dt\, \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)$$

and $\delta S = 0$ for all δq, if:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = 0} \quad - \left\{ \begin{array}{l} \text{Lagrange's} \\ \text{Equation} \end{array} \right.$$

Observe:

- Lagrange's equations determine trajectory $q(t)$. I.C.'s will be needed for solution.

- L.E. are PDE

- Lagrangian invariant to df/dt addition (i.e. undetermined to $+df/dt$).

i.e.
$$S = \int dt\, L \rightarrow \int dt\, \left(L + df/dt\right)$$
$$= S_0 + f(q(t_2)) - f(q(t_1))$$
$$\delta S = \delta S_0 + \delta(\Delta f) \qquad \frac{\partial f}{\partial q}\,\delta q\Big|_{t_2,t_1}$$

but $\delta q(t_{2,1}) = 0$

$$\delta S = \delta S_0$$ $\rightarrow$ no change in trajectory or physics.

Some obvious questions:

- is $q(t)$ obtained so a <u>minimum</u> of S?

  ⇒ Well, not really.... but is an extremum.

- what is L?

N.B. We know $L = T - V$

$T \to$ {kinetic energy} $V \to$ {potential energy}

but what if we didn't?

⇒ Structure of L:

- generally, by <u>symmetry</u>

i.e. consider <u>free</u> particle:

- here <u>non-relativistic</u>

- space-time <u>homogeneity</u> ⇒ L cannot depend on $\underline{x}, t$; only on $\underline{v}$

- space-time <u>isotropy</u> ⇒ L depends on $v^2 = \underline{v} \cdot \underline{v}$, only (not $\underline{v}$ → has direction content

$\therefore \quad L = L(v^2)$

Aside: why $L = L(v^2)$ and
$\partial L / \partial v^2 = m/2$? why not $L(v^2) \sim \alpha (v^2)^2$?

$\Rightarrow$ Principle of (Galilean) Relativity:

$\Rightarrow$

For two frames of reference related by infinitesimal Galilean boost, trajectories must be same.

$\left\{ \begin{array}{l} r = r' + Vt \\ t = t' \quad v = v' + V \\ V = \text{const.} \end{array} \right.$

Approach: show $L((v+dv)^2)$ and $L(v^2)$ differ by df/dt if [redacted] $\partial L / \partial v^2 = \text{const.}$

Check: $L\left[(v + dv)^2\right] - L(v^2)$

$$\cong L(v^2) + (2\underline{v} \cdot d\underline{v}) \left(\frac{\partial L}{\partial v^2}\right) + \text{h.o.}$$

$$\cancel{dv^2 \left(\frac{\partial L}{\partial v^2}\right)} - L(v^2)$$

$$= (2\underline{v} \cdot d\underline{v}) \frac{\partial L}{\partial v^2}$$

Now, if $\partial L/\partial v^2$ indep v^2 – i.e. constant:

$$L((\underline{v}+\delta\underline{v})^2) - L(v^2) \cong m\underline{v}\cdot\delta\underline{v}$$

$\delta\underline{v}$ fixed parameter, so:

$$\cong \frac{d}{dt}\left(m\,\underline{x}\cdot\delta\underline{v}\right)$$

$$= df/dt \quad \rightarrow \text{irrelevant.}$$

so $\frac{\partial L}{\partial v^2} = \text{const} = \frac{m}{2}$, by correspondence.

Note: $m > 0$ for minimum in S.

Thus: for free particle

$$L = \tfrac{1}{2} m v^2,$$ by symmetry and Galilean relativity

↳ kinetic energy).

⇒ Newton's 1st Law.

N.B. To show:

$$\Delta L = \frac{m}{2}(\underline{v}+\underline{V})^2 - \frac{mv^2}{2}$$

$$= \cancel{\frac{m v^2}{2}} + m\underline{v}\cdot\underline{V} + \frac{1}{2}m\underline{V}^2 - \cancel{\frac{m v^2}{2}}$$

$$= \frac{d}{dt}\left(m\underline{x}\cdot\underline{V} + \frac{1}{2}m\underline{V}^2 t\right)$$

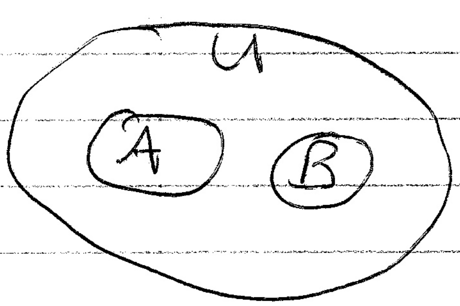
$$\downarrow$$

$$f$$

So : $L_{\substack{\text{free}\\\text{particle}}} = mv^2/2$ → see 8a

- For interacting particles, i.e. not free :

→ useful to introduce concept of open, closed system.



$U \equiv$ universe

A, B → systems

systems :

closed → non-interacting

open → interacting

if U formed by two, closed subsystems A, B (i.e. 2 free particles)

K.E. for standard coords:

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad , \quad \text{Cartesian}$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2), \quad \text{Cylindrical}$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin\theta^2 \dot{\phi}^2),$$
spherical

n.b.:

$$d\ell^2 = dx^2 + dy^2 + dz^2$$

$$d\ell^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$L_u = L_A + L_B \Rightarrow$

Lagrangians for closed sub-systems additive $\Rightarrow$ i.e. L for 2 systems together must asymptote to that for sum of L's for individual systems, at large separation.

[$\rightarrow$ Consider 2 particles $\Rightarrow$ must go to 2 indiv. free particles.]

Now, in non-relativistic limit:

For system of interacting particles which is closed, Lagrangian can be written as:

$$L = \sum_i \frac{m_i v_i^2}{2} + Q(\underline{r}_1, \underline{r}_2 \ldots)$$

interaction potential $\Leftrightarrow$ function of coordinates only

n.b.: – $v \ll c$ $\quad (c \to \infty)$

$\equiv$ $\Rightarrow$ particle "feels" effect of neighbor at retarded time

– $r\left(t - \frac{|\Delta r|}{c}\right) \to r(t) \Rightarrow$ instantaneous $\underset{c\to\infty}{}$

$$\frac{d}{dt} P_1 - \frac{\partial Q}{\partial r_1}(r_1, r_2, t) = 0$$

$$r_2 = r_2\left(t - \frac{|r_1 - r_2|}{c}\right) \xrightarrow[c \to \infty]{} r_2(t)$$

→ Now, in event that $q_i = x_i$ (generalized coordinates are Cartesian coordinates),

know Lagranges equations <u>must</u> reduce to Newton's Laws.

(Correspondence argument for Q, L).

L. E. :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) - \frac{\partial Q}{\partial \underline{x}_i} = 0$$

so

$$\frac{d}{dt}\left(\underline{p}_i\right) - \frac{\partial Q}{\partial \underline{x}_i} = 0$$

so $Q = -V$

$$\boxed{L = T - U}$$

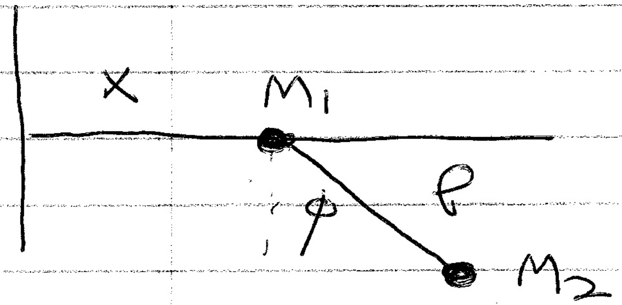
also natural to coin terminology:

$q_i \equiv$ generalized coordinate

$p_i = \frac{\partial L}{\partial \dot{q}_i} \equiv$ generalized momentum

Examples:

i) (Trivial)



Pendulum attached to freely sliding m_1.

computing energies:

$m_1 \Rightarrow T_1 = \frac{1}{2} m_1 \dot{x}^2$ GC: x, ϕ

$U = 0$

$m_2 \Rightarrow T_2 = \frac{1}{2} m_2 \left[\dot{x}^2 + \dot{y}^2\right]$

$= \frac{1}{2} m_2 \left[\left(\dot{(x + \ell\sin\phi)}\right)^2 + \left(\dot{(\ell\cos\phi)}\right)^2\right]$

$U = mg\ell(1 - \cos\phi)$

$$L = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \Big[\dot{x}^2 + 2\dot{x} \ell \dot{\phi} \cos\phi$$

$$+ \ell^2 \cos^2\phi \, \dot{\phi}^2 + \ell^2 \sin^2\phi \, \dot{\phi}^2 \Big]$$

$$- \, mg\ell \, (1 - \cos\phi)$$

$$L = \tfrac{1}{2} (m_1 + m_2) \dot{x}^2 + m_2 \Big[\frac{\ell^2}{2} \dot{\phi}^2 + \underbrace{\dot{x} \ell \dot{\phi} \cos\phi}_{\text{coupling}} \Big]$$

$$+ \, mg\ell \cos\phi \; \cancel{- mg\ell}$$

and EOM for x, ϕ follow.